SOME GENERALIZATIONS IN THE THEORY OF SUMMABLE DIVERGENT SERIES

BY

LLOYD LEROY SMAIL,

DISSERTATION

Submitted in Partial Fulfilment of the Requirements for the Degree of Doctor of Philosophy, in the Faculty of Pure Science, Columbia University

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INTRODUCTION.

In this paper, a process is given which leads to four general methods of summation of divergent series, and each of these methods includes as special cases several of the known methods. These latter are sufficiently indicated with their connections in the course of the discussion. It is shown that, in accordance with these general methods, every convergent series is summable and the generalized sum is equal to the ordinary sum; whilst a properly divergent series is not summable by these methods. Uniform summability, and the continuity of uniformly summable series and their term by term integration and differentiation, are discussed. Of the general theorems obtained, applications are made to the particular methods of Cesàro, Riesz, Borel, LeRoy, and the so-called Cesàro-Riesz methods of Hardy and Chapman. The methods of proof employed throughout are simpler than those hitherto used. In this way the essential properties of the various known methods are brought out, and greater uniformity of treatment is secured.

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SOME GENERALIZATIONS IN THE THEORY OF SUMMABLE DIVERGENT SERIES

CHAPTER I.

The Problem of Divergent Series.

1. Let us first seek what meaning can be assigned to an infinite series. In the first place we are to understand by an infinite series a symbol such as

$$(1) \qquad a_0 + a_1 + a_2 + \cdots + a_n + \cdots \equiv \sum_{n=0}^{\infty} a_n .$$

To assign a meaning to this symbol, the simplest and most natural way is to form the expression

$$s_n = \sum_{v=0}^{n} a_v ,$$

and when $\lim_{n \to \infty} s_n$ exists, to take this value, the so-called *sum* of the series, as a substitute for the series wherever the series occurs in calculations. Thus the method of convergent series is simply a particular method of associating a definite *number* with the series, and using this number in place of the series in calculations. This limit, $\lim_{n \to \infty} s_n$, however, only exists for certain series, while there are many series, the so-called *divergent* series, for which this limit does not exist. In order to be able to use such series, we must then find some method by which we can associate a definite number with the series, so that we can use this number in place of the series in calculations.

Our problem of divergent series is then to associate with such a series a *number,* which we call the *Sum** of the series, which should be defined in such a way that the resulting laws of calculation agree as far as possible with those of convergent series. If the series has variable terms, we wish to associate with it, not a number, but a *function,* which shall satisfy the above condition. Any definite method by which we can associate with a given series a Sum is called a method of summation.

2. Chapman† has stated a "general principle of summability" for any

* For convenience, we write the sum of a convergent series, in the usual sense, with small s, while this associated number here referred to, we write *Sum*, with capital *S*.

† *Quarterly Journ. of Math.*, Vol. 43, p. 4.

infinite form as follows: "When the sequence of finite forms, which defines or generates an infinite form, does not tend to a limit as the variables tend to infinity in the assigned order through the sequences of values constituting the domains of these variables, then we may agree that the number represented by the given infinite form is to be the limit of a sequence of associated finite forms, different from the members of the original sequence; the second sequence must of course be judiciously chosen, so that the limit to which it tends is usefully related to the original sequence. The number of its variables may be the same or greater than the original number; and the additional variables, if any, may or may not be required to tend to infinity."

Applying this to infinite series, we shall form sequences involving the terms of the given series, but which have limits even when the sequence $s_0, s_1, s_2, \cdots, s_n, \cdots$ has no limit. We shall not only form subsidiary sequences depending upon one index, but also sequences depending upon two indices (double sequences), and have simple and double limits of such sequences. An infinite series will then be said to be *summable* by any particular method if the corresponding subsidiary sequence has a limit.

If the terms of the given series are functions of a variable u, then this series will be said to be *uniformly summable* with respect to u in an interval (a, b), if the subsidiary sequence converges to its limit uniformly with respect to u in that interval.

3. Methods of summation have been classified by CHAPMAN* into *parametric* and *non-parametric*. A method of summation is parametric if the associated sequence used to determine the Sum contains a parameter of some kind upon the values of which depend whether or not the series is summable. If the associated sequence contains no such parameter, the method is called non-parametric.

4. As already explained, the Sum given by a method of summation is to be used whenever the given series is such that s_n has no limit. Now s_n has a limit for convergent series (in which case s_n has a finite limit) and for the properly divergent series (where s_n has the limit $+\infty$ or $-\infty$); our methods of summation are then designed primarily for the *oscillating* series. But convergent series and oscillating series may occur in the same piece of work, and so our methods of summation should be such that they will apply to convergent series and give a Sum equal to the ordinary sum. Likewise, when the methods of summation are applied to a properly divergent series, they should give a Sum equal to $+\infty$ or $-\infty$, according as s_n has the limit $+\infty$ or $-\infty$.

We formulate these requirements in the following conditions, which we call the *conditions of consistency*:

(1) When the given series is convergent, the Sum given by any method of summation must exist and coincide with the sum. Moreover, if the method

* *Quarterly Journ. of Math.*, Vol. 43, p. 8.

is parametric, the Sum must exist and coincide with the sum for all values of the parameter for which the method is applicable;

(2) When the given series is properly divergent, the Sum given by any method of summation must be $+\infty$ or $-\infty$, according as $\lim_{n\to\infty} s_n$ is $+\infty$ or $-\infty$;

(3) When the given series is uniformly convergent, it must be uniformly summable; and this must be so for every value of the parameter if the method is parametric.

CHAPTER II.

Some General Methods of Summation.

5. Having given a divergent series $\sum_0^\infty a_n$, we start with the expression

$$\sum_{v=0}^{[n]} a_v f_v, \tag{2}$$

where n is always taken positive, and $[n]$ denotes the greatest integer $\overline{\gtrless} n$, and f_v is a function of certain variables and parameters to be specified presently. We shall study the limiting properties of this expression, and find out under what circumstances these limits will satisfy the conditions of consistency.

We first enunciate some of the cases which may occur and which we shall study:

I. f_v may be a function of n and also of a parameter k; then, keeping k fixed, we seek the simple limit

$$\lim_{n\to\infty} \sum_{v=0}^{[n]} a_v f_v(n, k). \tag{3}$$

II. f_v may be a function of a variable x; we then seek the repeated double limits

$$\lim_{n\to\infty} \lim_{x\to\infty} \sum_{v=0}^{n} a_v f_v(x),\text{*} \tag{4}$$

$$\lim_{x\to\infty} \lim_{n\to\infty} \sum_{v=0}^{n} a_v f_v(x), \tag{5}$$

or the Pringsheim double limit

$$\lim_{n,\, x\to\infty} \sum_{v=0}^{n} a_v f_v(x), \tag{6}$$

where n, x tend to ∞ independently but simultaneously.

III. f_v may be a function of two variables n and p; we then seek the repeated double limits

$$\lim_{n\to\infty} \lim_{p\to\infty} \sum_{v=0}^{[n]} a_v f_v(n, p), \tag{7}$$

$$\lim_{p\to\infty} \lim_{n\to\infty} \sum_{v=0}^{[n]} a_v f_v(n, p), \tag{8}$$

* The case $x \to \infty$ may be regarded as a general case, since any other, $x \to a$, can be reduced to this by transformation.

or we may seek the Pringsheim double limit

(9) $$\lim_{n,\,p\to\infty} \sum_{v=0}^{[n]} a_v f_v(n,\,p),$$

or we may seek the double limit along a path F

(10) $$\lim_{\substack{n,\,p\to\infty \\ F}} \sum_{v=0}^{[n]} a_v f_v(n,\,p),$$

where n, p tend to ∞ simultaneously though not independently, but in such a way that a functional relation

$$F(n,\,p) = 0$$

holds between them.*

In place of starting with the expression (2), we may start with the expression

(11) $$\sum_{v=0}^{n} s_v f_v,$$

where $s_v = \sum_{i=0}^{v} a_i$. Then we have the case

IV. f_v may be a function of x, and we seek the repeated double limits

(12) $$\lim_{n\to\infty} \lim_{x\to\infty} \sum_{v=0}^{n} s_v f_v(x),$$

(13) $$\lim_{x\to\infty} \lim_{n\to\infty} \sum_{v=0}^{n} s_v f_v(x).$$

Let us now examine each of these cases more in detail.

Case I.

6. Let $f_v(n,\,k)$ be defined for all positive integral values of v, including 0, for all positive values of n, and for a certain range of real values of the parameter k.

Suppose further that $f_v(n,\,k)$ satisfies the following conditions:

1° $$0 \overline{\gtrless} f_v(n,\,k) \overline{\gtrless} 1 \qquad \text{for every } v,\, n,\, k;$$

2° when n and k are fixed, the sequence

$$f_0,\, f_1,\, f_2,\, \cdots,\, f_v,\, \cdots$$

is monotonic decreasing;

3° $$\lim_{n\to\infty} f_v(n,\,k) = 1 \qquad \text{for } v \text{ fixed};$$

4° $$f_0(n,\,k) = 1;$$

5° $$\lim_{n\to\infty} f_{[n]}(n,\,k) = 0.$$

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* See Hardy and Chapman, *Quarterly Journ. of Math.*, Vol. 42, p. 187.

When limit (3) exists and has the value S:

$$(3') \qquad \lim_{n \to \infty} \sum_{v=0}^{[n]} a_v f_v(n, k) = S,$$

then we shall say that the series $\sum_0^\infty a_v$ is *summable* (I) with *Sum* S.

We now proceed to show that this method of summation satisfies the conditions of consistency. It may be noted that this is a parametric method.

7. THEOREM 1. *If the series* $\sum_0^\infty a_v$ *is convergent with sum* s, *then*

$$(3) \qquad \lim_{n \to \infty} \sum_{v=0}^{[n]} a_v f_v(n, k)$$

exists and is equal to s *for every value of* k *for which* f_v *is defined; so that every convergent series is summable* (I) *with Sum equal to sum, and part* (1) *of the conditions of consistency is satisfied.*

Put

$$1 - f_v(n, k) = g_v(n, k),$$

then by condition 1°, we have

$$0 \leqq g_v(n, k) \leqq 1,$$

by 2° the sequence $g_0, g_1, g_2, \cdots, g_v, \cdots$ is monotonic, and by 3°,

$$\lim_{n \to \infty} g_v(n, k) = 0.$$

Put

$$G(n, k) \equiv \sum_{v=0}^{[n]} a_v g_v(n, k) = \sum_{v=0}^{[n]} a_v - \sum_{v=0}^{[n]} a_v f_v(n, k).$$

Then we have to prove that

$$(a) \qquad \lim_{n \to \infty} G(n, k) = 0.$$

We may write

$$(b) \qquad G(n, k) = \sum_{v=0}^{N} a_v g_v + \sum_{v=N+1}^{[n]} a_v g_v.$$

Let us consider first the second sigma of (b). By ABEL's lemma,*

$$\left| \sum_{v=N+1}^{[n]} a_v g_v(n, k) \right| \leqq Ag \leqq A,$$

where A is the upper bound of

$$\left| \sum_{v=N+1}^{r} a_v \right|$$

* See BROMWICH, Theory of Infinite Series, § 23.

for

$$r = N + 1,\ N + 2,\ \cdots,\ [n],$$

and g is the upper bound of the g_v for

$$v = N + 1,\ \cdots,\ [n];$$

by condition 1°, $g \leqq 1$. Now since Σa_v is convergent, we can choose N_0 so that $A < \epsilon / 2$ for $N > N_0$, where ϵ is any arbitrarily small positive number. Hence

$$(c) \qquad \left| \sum_{v=N+1}^{[n]} a_v g_v \right| < \frac{\epsilon}{2}$$

for $N > N_0$, $n > N + 1$.

Now consider the first sigma of (*b*). By condition 3°, when N is once fixed, for every given positive number ϵ', we can find an integer m_v such that for $n > m_v$,

$$g_v(n, k) < \epsilon' \qquad \text{for } v = 0,\ 1,\ 2,\ \cdots,\ N.$$

Let m be the greatest of the finite set of integers $m_0, m_1, \cdots, m_N$; then for $n > m$, each $g_v(n, k) < \epsilon'$ ($v = 0, \cdots, N$).

$$\therefore \left| \sum_{v=0}^{N} a_v g_v(n, k) \right| \leqq \sum_{v=0}^{N} | a_v | \cdot g_v(n, k) < \epsilon' \sum_{v=0}^{N} | a_v | \quad \text{for } n > m.$$

Choose

$$\epsilon' < \frac{\epsilon}{2 \sum_{v=0}^{N} | a_v |},$$

then

$$(d) \qquad \left| \sum_{v=0}^{N} a_v g_v \right| < \frac{\epsilon}{2} \qquad \text{for } n > m.$$

From (*c*) and (*d*), we now see that

$$| G(n, k) | < \epsilon \qquad \text{for } n > N',$$

where N' is $\geqq N_0$ or m.

Hence

$$\lim_{n \to \infty} \sum_{v=0}^{[n]} a_v f_v(n, k) = \lim_{n \to \infty} \sum_{v=0}^{[n]} a_v = s.$$

The above argument has not explicitly involved the value of k, so that the theorem is true for every k for which f_v satisfies the conditions of § 6.

It may be remarked that all the conditions of § 6 are not required for this proof; we need only conditions 1°, 3°, and 2° can be replaced by the requirement that the sequence (f_v) be monotonic.

8. Theorem 2. *If the series* $\sum_0^{\infty} a_v(u)$ *is uniformly convergent with respect to* u *in the closed interval* (a, b), *with sum* $s(u)$, *then* $\sum_{v=0}^{[n]} a_v(u) f_v(n, k)$ *tends*

to its limit $s(u)$ uniformly with respect to u in the interval (a, b); so that a uniformly convergent series is also uniformly summable (I), *and part* (3) *of the conditions of consistency is satisfied.*

Using the notation of § 7, we must prove that $G(n, k, u) \to 0$ uniformly in (a, b). We find, as before,

$$\left| \sum_{v=N+1}^{[n]} a_v(u) \cdot g_v(n, k) \right| \overline{\overline{<}} A(u) \cdot g \overline{\overline{<}} A(u),$$

but since $\Sigma a_v(u)$ is uniformly convergent, we can find N_0 such that $A(u) < \epsilon / 2$ for $N > N_0$ and for every u in (a, b). We then have

$$(c) \qquad \left| \sum_{v=N+1}^{[n]} a_v(u) g_v(n, k) \right| < \frac{\epsilon}{2}$$

for $N > N_0$ and for every u in (a, b).

As before, we find

$$\left| \sum_{v=0}^{N} a_v(u) \cdot g_v(n, k) \right| < \epsilon' \sum_{v=0}^{N} | a_v(u) | \qquad \text{for } n > m.$$

Let K be the upper bound of

$$\sum_{v=0}^{N} | a_v(u) |$$

for u in (a, b); take

$$\epsilon' < \frac{\epsilon}{2K},$$

then

$$(d) \qquad \left| \sum_{v=0}^{N} a_v(u) \cdot g_v(n, k) \right| < \frac{\epsilon}{2} \qquad \text{for } n > m,$$

and for every u in (a, b).

Hence $G(n, k, u) \to 0$ uniformly with respect to u, and our theorem follows.

9. THEOREM 3. *If the series $\sum_{0}^{\infty} a_v$ is properly divergent, so that*

$$\lim_{n \to \infty} s_n = +\infty,$$

then

$$\lim_{n \to \infty} \sum_{v=0}^{[n]} a_v f_v(n, k) = +\infty;$$

hence a properly divergent series is not summable (I) *with finite Sum, and part* (2) *of the conditions of consistency is satisfied.*

We have

$$\sum_{v=0}^{[n]} a_v f_v = s_0 f_0 + (s_1 - s_0) f_1 + (s_2 - s_1) f_2 + \cdots + (s_{[n]} - s_{[n]-1}) f_{[n]}$$
$$= s_0 (f_0 - f_1) + s_1 (f_1 - f_2) + \cdots + s_{[n]-1} (f_{[n]-1} - f_{[n]}) + s_{[n]} f_{[n]} .$$

(14) $$\therefore \sum_{v=0}^{[n]} a_v f_v = \sum_{v=0}^{[n]-1} s_v (f_v - f_{v+1}) + s_{[n]} f_{[n]} .$$

Since

$$\lim_{n\to\infty} s_n = + \infty ,$$

if K is any arbitrarily large number, we can find m such that we can put

$$s_v = K + r_v \qquad \text{for } v > m, \text{ where } r_v \geqq 0 .$$

Put

$$f_v - f_{v+1} \equiv h_v .$$

Then (14) becomes

$$\sum_{v=0}^{[n]} a_v f_v (n, k) = \sum_{v=0}^{m} s_v h_v (n, k) + K \sum_{v=m+1}^{[n]-1} h_v (n, k)$$
$$+ \sum_{v=m+1}^{[n]-1} r_v h_v (n, k) + s_{[n]} f_{[n]} (n, k) = \sum_{v=0}^{m} (s_v - K) h_v (n, k)$$
$$+ K \sum_{v=0}^{[n]-1} h_v (n, k) + \sum_{v=m+1}^{[n]-1} r_v h_v (n, k) + s_{[n]} f_{[n]} (n, k) .$$

But

$$\sum_{v=0}^{[n]-1} h_v (n, k) = (f_0 - f_1) + (f_1 - f_2) + \cdots + (f_{[n]-1} - f_{[n]}) = f_0 - f_{[n]} .$$

(a) $$\therefore \sum_{v=0}^{[n]} a_v f_v (n, k) = \sum_{v=0}^{m} (s_v - K) h_v (n, k) + K \{ f_0 (n, k) - f_{[n]} (n, k) \}$$
$$+ \sum_{v=m+1}^{[n]-1} r_v h_v (n, k) + s_{[n]} f_{[n]} (n, k) .$$

By condition 2°,

$$h_v (n, k) \text{ is } \geqq 0;$$

by 3°,

$$\lim_{n\to\infty} h_v (n, k) = 0;$$

by 1°,

$$f_{[n]} (n, k) \geqq 0 .$$

Using these results, and conditions 4°, 5°, we get

$$\lim_{n\to\infty} \sum_{v=0}^{[n]} a_v f_v (n, k) = K + \lim_{n\to\infty} \sum_{v=m+1}^{[n]-1} r_v h_v (n, k) + \lim_{n\to\infty} s_{[n]} f_{[n]} (n, k) .$$
$$\geqq K .$$

Since K is arbitrarily large, our theorem follows at once.

For this proof, all the conditions 1°–5° of § 6 are required.

10. THEOREM 4. *If the series* $\sum_{v=0}^{\infty} a_v(u)$ *is uniformly summable* (I) *with respect to* u *in an interval* (a, b), *with Sum* $S(u)$, *and if the terms* $a_v(u)$ *are continuous functions of* u *in* (a, b), *then* $S(u)$ *is a continuous function of* u *in* (a, b).

Since

$$S(u) = \lim_{n \to \infty} \sum_{v=0}^{[n]} a_v(u) f_v(n, k) = \sum_{0}^{\infty} a_v(u) f_v(n, k),$$

it follows that $S(u)$ is the sum of a uniformly convergent series of continuous functions, and hence, by the properties of uniformly convergent series, $S(u)$ is continuous.

The theorem may be proved directly as follows:*

Put

$$\sum_{v=0}^{[n]} a_v(u) f_v(n, k) \equiv S_n(u).$$

We have

$$|S(u+h) - S(u)| \leqq |S(u+h) - S_n(u+h)| + |S_n(u+h) - S_n(u)| + |S_n(u) - S(u)|.$$

Now since

$$\lim_{n \to \infty} S_n(u) = S(u)$$

and the approach is uniform with respect to u in (a, b), we can find N such that for $n > N$,

$$|S(u) - S_n(u)| < \frac{\epsilon}{3},$$

and

$$|S(u+h) - S_n(u+h)| < \frac{\epsilon}{3},$$

where ϵ is any arbitrarily small positive number.

Again, since $S_n(u)$ is a sum of a finite number of continuous functions, we can find δ such that

$$|S_n(u+h) - S_n(u)| < \frac{\epsilon}{3} \qquad \text{for } |h| < \delta.$$

Hence we obtain

$$|S(u+h) - S(u)| < \epsilon \qquad \text{for } |h| < \delta,$$

which shows that $S(u)$ is continuous.

11. THEOREM 5. *If the series* $\sum_{0}^{\infty} a_v(u)$ *is uniformly summable* (I) *with respect to* u *in the closed interval* (a, b), *with Sum* $S(u)$, *and if* (c_1, c_2) *is*

* See CHAPMAN, *Quarterly Journ. of Math.*, Vol. 43, p. 12.

an interval contained in (a, b), *and if the terms* $a_v(u)$ *are integrable in* (c_1, c_2), *then the series obtained by integrating the given series term by term over the range* (c_1, c_2) *is summable* (I), *and its Sum is* $\int_{c_1}^{c_2} S(u)\,du$.

We can write

$$S(u) = S_n(u) + \delta_n(u),$$

where $\delta_n(u)$ is such that we can find N so that for $n > N$,

$$|\delta_n(u)| < \epsilon,$$

however small ϵ may be, for every u in (a, b) and

$$S_n(u) = \sum_{v=0}^{[n]} a_v(u) f_v(n, k).$$

$$\therefore \int_{c_1}^{c_2} S(u)\,du = \int_{c_1}^{c_2} S_n(u)\,du + \int_{c_1}^{c_2} \delta_n(u)\,du$$

$$= \sum_{v=0}^{[n]} f_v(n, k) \cdot \int_{c_1}^{c_2} a_v(u)\,du + \int_{c_1}^{c_2} \delta_n(u)\,du.$$

But

$$\left|\int_{c_1}^{c_2} \delta_n(u)\,du\right| \leqq \int_{c_1}^{c_2} |\delta_n(u)| \cdot |du| < \epsilon(c_2 - c_1), \quad (n > N),$$

$$\therefore \lim_{n\to\infty} \int_{c_1}^{c_2} \delta_n(u)\,du = 0,$$

and we have

$$\lim_{n\to\infty} \sum_{v=0}^{[n]} f_v(n, k) \int_{c_1}^{c_2} a_v(u)\,du = \int_{c_1}^{c_2} S(u)\,du.$$

Hence the series $\sum_0^\infty \int_{c_1}^{c_2} a_v(u)\,du$ is summable (I) with Sum $\int_{c_1}^{c_2} S(u)\,du$.*

Observe that this proof holds for any function $f_v(n, k)$, such that $S_n(u)$ approaches a limit $S(u)$ uniformly, even though it should fail to satisfy all the conditions 1°–5° of § 6.

12. Theorem 6. *If the series* $\sum_0^\infty a_v(u)$ *is summable* (I) *with Sum* $S(u)$, *then if the series* $\sum_0^\infty a'_v(u)$ *obtained by differentiating term by term the given series is uniformly summable* (I) *with respect to* u *in an interval* (a, b), *with Sum* $\sigma(u)$, *we have*

$$\sigma(u) = \frac{d}{du} S(u).$$

* See Chapman, *Quarterly Journ. of Math.*, Vol. 43, pp. 12–13.

By the preceding theorem, the series

$$\sum_{0}^{\infty} \int_{c_1}^{u} a_v'(u)\,du,$$

where c_1 and u are in the interval (a, b), is summable (I) with Sum

$$\int_{c_1}^{u} \overset{\sigma}{\cancel{S}}(u)\,du.$$

That is,

$$\lim_{n \to \infty} \sum_{v=0}^{[n]} f_v(n, k) \int_{c_1}^{u} a_v'(u)\,du = \int_{c_1}^{u} \sigma(u)\,du;$$

or since

$$\int_{c_1}^{u} a_v'(u)\,du = a_v(u) - a_v(c_1),$$

we have

$$\lim_{n \to \infty} \sum_{v=0}^{[n]} \{a_v(u) - a_v(c_1)\} f_v(n, k) = \int_{c_1}^{u} \sigma(u)\,du. \tag{a}$$

But since $\sum_{0}^{\infty} a_v(u)$ is summable (I) with Sum $S(u)$, the left hand side of (a) is equal to $S(u) - S(c_1)$, or

$$\int_{c_1}^{u} \sigma(u)\,du = S(u) - S(c_1). \tag{b}$$

This equation shows that

$$\sigma(u) = \frac{d}{du} S(u).$$

Case II.

13. Let $f_v(x)$ be defined for every positive value of x, and for $v = 0, 1, 2, \cdots$.

Suppose further that $f_v(x)$ satisfies the following conditions:

1° $\quad 0 \leqq f_v(x) \leqq 1 \quad$ for every v, x;

2° when x is fixed, the sequence $f_0, f_1, \cdots, f_v, \cdots$ is decreasing;

3° $\quad \lim\limits_{x \to \infty} f_v(x) = 1 \quad$ for v fixed;

4° $\quad \lim\limits_{x \to \infty} \lim\limits_{n \to \infty} f_n(x) = 0.$

14. We shall first show that limits (4) and (6) cannot give rise to methods of summation of non-convergent series.

We have, by condition 3°,

$$\lim_{x \to \infty} \sum_{v=0}^{n} a_v f_v(x) = \sum_{v=0}^{n} a_v,$$

and

$$\lim_{n\to\infty}\lim_{x\to\infty}\sum_{v=0}^{n} a_v f_v(x) = \lim_{n\to\infty}\sum_{v=0}^{n} a_v .$$

Hence, limit (4) can only exist when the series $\sum_0^\infty a_v$ is convergent, and will not give a method of summation of non-convergent series. We need then consider limit (4) no further.

From the well-known theorem:*

"*If* $\lim_{p,q\to\infty} a_{p,q}$ *exists and if* $\lim_{p\to\infty} a_{p,q}$ *exists for every* q, *then*

$$\lim_{q\to\infty}(\lim_{p\to\infty} a_{p,q}) = \lim_{p,q\to\infty} a_{p,q},\text{''}$$

we see that if $\lim_{p\to\infty} a_{p,q}$ exists for every q, then $\lim_{p,q\to\infty} a_{p,q}$ can only exist when $\lim_{q\to\infty}\lim_{p\to\infty} a_{p,q}$ exists.

Then since $\lim_{x\to\infty}\sum_{v=0}^{n} a_v f_v(x)$ exists for every n, it follows that the Pringsheim double limit

$$\lim_{n,x\to\infty}\sum_{v=0}^{n} a_v f_v(x) \tag{6}$$

can only exist when the repeated double limit

$$\lim_{n\to\infty}\lim_{x\to\infty}\sum_{v=0}^{n} a_v f_v(x) \tag{4}$$

exists, i. e., only when the series Σa_v is convergent. Hence the limit (6) will not apply to non-convergent series, and we need not consider it further.

15. When the limit (5) exists and is equal to S,

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_{v=0}^{n} a_v f_v(x) = S, \tag{5'}$$

we shall say that the series Σa_v is *summable* (*II*) with Sum S.

We shall now show that this method of summation satisfies the conditions of consistency.

16. Theorem 7. *If the series* $\sum_0^\infty a_v$ *is convergent with sum* s, *then*

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_{v=0}^{n} a_v f_v(x) \tag{5}$$

exists and is equal to s; *that is, every convergent series is summable* (II) *with Sum equal to sum, and so part* (1) *of the conditions of consistency is satisfied.*

* See Nielsen, Lehrbuch der unendlichen Reihen, p. 76.

Put

$$1 - f_v(x) = g_v(x);$$

then from conditions 1°, 2°, 3°, it follows that

$$0 \leqq g_v(x) \leqq 1$$

for every v, x, that the sequence $g_0, g_1, \cdots, g_v, \cdots$ is monotonic, and that

$$\lim_{x \to \infty} g_v(x) = 0$$

for v fixed.

Put

$$G(n, x) = \sum_{v=0}^{n} a_v g_v(x),$$

then since

$$G(n, x) = \sum_{0}^{n} a_v - \sum_{0}^{n} a_v f_v(x),$$

we must prove that

$$\lim_{x \to \infty} \lim_{n \to \infty} G(n, x) = 0.$$

We shall first show that the Pringsheim double limit

$$\lim_{n, x \to \infty} G(n, x) = 0.$$

Writing $G(n, x)$ in the form

$$G(n, x) = \sum_{v=0}^{N} a_v g_v(x) + \sum_{v=N+1}^{n} a_v g_v(x),$$

we first apply ABEL's lemma to the second sigma, and obtain

$$\left| \sum_{v=N+1}^{n} a_v g_v(x) \right| \leqq A \cdot g \leqq A \qquad \text{for every } x,$$

where A is the upper bound of

$$\left| \sum_{N+1}^{r} a_r \right| \quad \text{for} \quad r = N + 1, \cdots, n,$$

and g is the upper bound of the terms g_v for

$$v = N + 1, \cdots, n,$$

so that $g \overline{\gtrless} 1$. Since Σa_v is convergent, we can choose N_0 so that, for every ϵ, for $N > N_0$, we have $A < \epsilon / 2$; hence

$$(a) \qquad \left| \sum_{v=N+1}^{n} a_v g_v(x) \right| < \frac{\epsilon}{2}$$

for $N > N_0$ and for every x.

Now keeping N fixed, for any given ϵ', by condition 3° we can find an integer X_v such that for $x > X_v$, we have $g_v(x) < \epsilon'$ for $v = 0, 1, \cdots, N$. Let X be the greatest of the finite set of integers X_v, $(v = 0, 1, \cdots, N)$, then for $x > X$, we have each $g_v(x) < \epsilon'$ $(v = 0, \cdots, N)$.

$$\therefore \left|\sum_{v=0}^{N} a_v\, g_v(x)\right| \leqq \sum_{v=0}^{N} |a_v| \cdot g_v(x) < \epsilon' \cdot \sum_{v=0}^{N} |a_v| \qquad \text{for } x > X.$$

If we take

$$\epsilon' < \frac{\epsilon}{2\sum_0^N |a_v|},$$

we get

$$(b) \qquad \left|\sum_{v=0}^{N} a_v\, g_v(x)\right| < \frac{\epsilon}{2} \qquad \text{for } x > X.$$

From (a) and (b), we now obtain

$$|G(n, x)| < \epsilon \qquad \text{for } n > N_0 + 1,\ x > X,$$

so that

$$\lim_{n,\, x \to \infty} G(n, x) = 0.$$

We now apply the theorem (already quoted in § 14):
"*If* $\lim\limits_{p,\, q \to \infty} a_{p,\, q}$ *exists and* $\lim\limits_{p \to \infty} a_{p,\, q}$ *exists for every* q, *then*

$$\lim_{q \to \infty} \lim_{p \to \infty} a_{p,\, q} = \lim_{p,\, q \to \infty} a_{p,\, q}."$$

Making use of the theorem:* *A convergent series remains convergent if its terms are each multiplied by factors which form a bounded monotonic sequence,* we see that the series $\sum_0^\infty a_v\, g_v(x)$ is convergent and $\lim\limits_{n \to \infty} G(n, x)$ exists. Hence by the theorem just quoted, we have

$$\lim_{x \to \infty} \lim_{n \to \infty} G(n, x) = 0,$$

and

$$\lim_{x \to \infty} \lim_{n \to \infty} \sum_{v=0}^{n} a_v\, f_v(x) = \lim_{n \to \infty} \sum_{v=0}^{n} a_v = s.$$

Not all the conditions of § 13 were used in this proof; we need only 1°, 3°, and in place of 2° we need only require the sequence (f_v) to be monotonic.

17. Theorem 8. *If the series* $\sum_0^\infty a_v$ *is properly divergent, so that*

$$\lim_{n \to \infty} s_n = +\infty,$$

* See Bromwich, Theory of Infinite Series, § 19.

then

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_{v=0}^{n} a_v f_v(x) = +\infty;$$

that is, no properly divergent series can be summable (II) *with finite Sum, and hence part* (2) *of the conditions of consistency is satisfied.*

Starting with equation (14) of § 9, and proceeding exactly as in § 9, we arrive at the equation corresponding to (a):

$$(a) \qquad \sum_{v=0}^{n} a_v f_v(x) = \sum_{v=0}^{m} (s_v - K)\, h_v(x) + K\{f_0(x) - f_n(x)\} + \sum_{v=m+1}^{n-1} r_v\, h_v(x) + s_n f_n(x),$$

where

$$h_v(x) = f_v(x) - f_{v+1}(x).$$

We find from the conditions of § 13, that

$$\lim_{x\to\infty} h_v(x) = 0, \qquad \lim_{x\to\infty} f_0(x) = 1, \qquad \lim_{x\to\infty}\lim_{n\to\infty} f_n(x) = 0.$$

$$\therefore \lim_{x\to\infty}\lim_{n\to\infty}\sum_{v=0}^{n} a_v f_v(x) = K + \lim_{x\to\infty}\lim_{n\to\infty}\sum_{v=0}^{n-1} r_v\, h_v(x) + \lim_{x\to\infty}\lim_{n\to\infty} s_n f_n(x).$$

But since $r_v > 0$, $h_v(x) > 0$, and $s_n > 0$, $f_n(x) \geqq 0$, we have

$$\lim_{x\to\infty}\lim_{n\to\infty}\sum_{v=0}^{n-1} r_v\, h_v(x) > 0, \qquad \lim_{x\to\infty}\lim_{n\to\infty} s_n f_n(x) > 0,$$

$$\therefore \lim_{x\to\infty}\lim_{n\to\infty}\sum_{v=0}^{n} a_v f_v(x) > K.$$

Since K can be taken as large as we please, our theorem follows at once.

18. Before proceeding to show that this method of summation satisfies part (3) of the conditions of consistency, it will be necessary to make the notion of uniform summability (II) more precise; for this notion involves uniform approach to a repeated double limit.

A definition of uniform approach to a Pringsheim double limit is easy to formulate. If $a_{p,q}(u)$ is a function of u, and if for each value of u in an interval (a, b), the Pringsheim double limit

$$\lim_{p,q\to\infty} a_{p,q}(u)$$

exists and is equal to $a(u)$, then we shall say that $a_{p,q}(u)$, approaches its Pringsheim double limit $a(u)$ *uniformly* with respect to u in the interval (a, b), if for any positive ϵ, two integers P, Q can be found such that for every

$p \geqq P$, $q \geqq Q$,

$$| a(u) - a_{p,q}(u) | < \epsilon$$

for every value of u in (a, b).

19. THEOREM 9. *If $a_{p,q}(u)$ approaches its Pringsheim double limit*

$$\lim_{p,q\to\infty} a_{p,q}(u) = a(u)$$

uniformly with respect to u in an interval (a, b), and if for each q, $a_{p,q}(u)$ approaches a simple limit

$$\lim_{p\to\infty} a_{p,q}(u) = a_q(u)$$

uniformly with respect to u in (a, b), then $a_q(u)$ approaches the simple limit

$$\lim_{q\to\infty} a_q(u) = a(u)$$

uniformly with respect to u in (a, b).

By hypothesis, we can find numbers P, Q such that

$$(a) \qquad | a(u) - a_{p,q}(u) | < \frac{\epsilon}{2}$$

for $p > P$, $q > Q$, and for every u in (a, b); also we can find a number P'_q (depending in general upon q) such that

$$(b) \qquad | a_{p,q}(u) - a_q(u) | < \frac{\epsilon}{2}$$

for $p > P'_q$ and for every u in (a, b). Adding (a) and (b), we get

$$| a(u) - a_q(u) | < \epsilon$$

for $q > Q$ and for every u in (a, b), which gives the result stated in the theorem.

20. The theorem just proved suggests a definition of uniform approach to a repeated double limit, which will be useful in our later discussion of uniform summability.

If for each value of u in the interval (a, b), the repeated double limit

$$\lim_{q\to\infty} \lim_{p\to\infty} a_{p,q}(u)$$

exists and is equal to $a(u)$, then we shall say that $a_{p,q}(u)$ approaches its repeated double limit $a(u)$ *uniformly* with respect to u in (a, b), if $a_{p,q}(u)$ approaches its simple limit

$$\lim_{p\to\infty} a_{p,q}(u) = a_q(u)$$

uniformly with respect to u in (a, b), and if $a_q(u)$ approaches its limit

$$\lim_{q\to\infty} a_q(u) = a(u)$$

uniformly with respect to u in (a, b).

Our definition of uniform summability (II) will then be: If $\sum_{v=0}^{n} a_v(u) f_v(x)$ approaches its repeated double limit

$$\lim_{x\to\infty} \lim_{n\to\infty} \sum_{v=0}^{n} a_v(u) f_v(x) = S(u)$$

uniformly with respect to u in an interval (a, b), then the series $\sum_0^\infty a_v(u)$ is said to be *uniformly summable* (II) with respect to u in (a, b).

21. We are now in a position to discuss the uniform summability (II) of a uniformly convergent series; but before entering upon that, we shall first prove an auxiliary theorem, in order not to interrupt the argument later.

THEOREM 10. *A uniformly convergent series* $\sum_0^\infty a_v(u)$ *remains uniformly convergent if its terms are each multiplied by factors* g_v, *provided that the sequence* (g_v) *is monotonic, and that* $|g_v| <$ *a constant* c.*

Since the sequence (g_v) is monotonic and $|g_v| < c$, g_v must approach a limit, call it g. Put $b_v = g - g_v$ when (g_v) is an increasing sequence, and $b_v = g_v - g$ when (g_v) is decreasing. Then the sequence (b_v) is monotonic decreasing and approaches the limit 0. Since

$$a_v(u) \cdot g_v = a_v(u) \cdot g - a_v(u) \cdot b_v \quad \text{or} \quad a_v(u) \cdot g + a_v(u) \cdot b_v,$$

we need only prove that $\sum_0^\infty a_v(u) \cdot b_v$ is uniformly convergent. If $A(u)$ is the upper bound of

$$\left| \sum_{m+1}^{r} a_v(u) \right|$$

for

$$r = m + 1, \cdots, m + p,$$

we have by ABEL's lemma

$$\left| \sum_{m+1}^{m+p} a_v(u) \cdot b_v \right| < A(u) \cdot b_{m+1} < A(u) \cdot b_0.$$

But since $\sum_0^\infty a_v(u)$ is uniformly convergent, we can find M such that for $m > M$, we have

$$A(u) < \frac{\epsilon}{b_0},$$

* This is a generalization of the theorem of § 19 in BROMWICH, Theory of Infinite Series.

however small ϵ may be, for every u in an interval (a, b).

$$\therefore \left| \sum_{m+1}^{m+p} a_v(u) \cdot b_v \right| < \epsilon$$

for $m > M$ and for every positive integer p, and for every u in (a, b). Hence the series $\sum_0^\infty a_v(u)\, b_v$ is uniformly convergent in (a, b).

22. THEOREM 11. *If the series* $\sum_0^\infty a_v(u)$ *is uniformly convergent with respect to* u *in the interval* (a, b), *with sum* $s(u)$, *then* $\sum_{v=0}^{n} a_v(u) \cdot f_v(x)$ *tends to its repeated double limit* $s(u)$ *uniformly with respect to* u *in* (a, b); *that is, a uniformly convergent series is also uniformly summable* (II), *and so part* (3) *of the conditions of consistency is satisfied.*

We shall first show that $\sum_{v=0}^{n} a_v(u) \cdot f_v(x)$ approaches its Pringsheim double limit $s(u)$ uniformly with respect to u in (a, b).

Using the notation of § 16, we have to prove that numbers N, X can be found such that for every $n > N$, $x > X$,

$$| G(n, x, u) | < \epsilon$$

for every u in (a, b).

As in § 16, we find by ABEL'S lemma

$$\left| \sum_{v=N+1}^{n} a_v(u)\, g_v(x) \right| \leqq A(u) \cdot g \leqq A(u),$$

for every x and for every u, where $A(u)$ is the upper bound of

$$\left| \sum_{N+1}^{r} a_v(u) \right| \quad \text{for} \quad r = N+1, \cdots, n,$$

and g is the upper bound of the terms g_v for $v = N+1, \cdots, n$. Since $\sum_0^\infty a_v(u)$ is uniformly convergent, we can find N_0 such that for $N > N_0$, we have $A(u) < \epsilon / 2$ for every u in (a, b).

(a) $$\therefore \left| \sum_{v=N+1}^{n} a_v(u)\, g_v(x) \right| < \frac{\epsilon}{2}$$

for $N > N_0$, for every x, and for every u in (a, b).

As before, (§ 16),

$$\left| \sum_{v=0}^{N} a_v(u)\, g_v(x) \right| < \epsilon' \sum_{v=0}^{N} | a_v(u) | \qquad \text{for every } x > X.$$

Let K be the upper bound of $\sum_0^N |a_v(u)|$ for u in (a, b); and take $\epsilon' < \frac{\epsilon}{2K}$. Then

$$(b) \qquad \left| \sum_{v=0}^{N} a_v(u)\, g_v(x) \right| < \frac{\epsilon}{2}$$

for $x > X$ and for every u in (a, b).

Hence

$$\left| \sum_{v=0}^{n} a_v(u)\, g_v(x) \right| < \epsilon$$

for $n > N > N_0$, $x > X$, and for every u in (a, b); that is, $G(n, x, u)$ approaches its Pringsheim double limit 0 uniformly.

$g_v(x)$ evidently satisfies the conditions of Theorem 10, so that, since $\Sigma a_v(u)$ is uniformly convergent, $\sum_0^\infty a_v(u) \cdot g_v(x)$ is also uniformly convergent; that is, the limit

$$\lim_{n \to \infty} \sum_{v=0}^{n} a_v(u)\, g_v(x)$$

is approached uniformly with respect to u in (a, b). Now applying Theorem 9, we see that the limit

$$\lim_{x \to \infty} \lim_{n \to \infty} \sum_{v=0}^{n} a_v(u)\, g_v(x)$$

is approached uniformly with respect to u in (a, b). Our theorem then follows.

23. THEOREM 12. *If the series $\sum_0^\infty a_v(u)$ is uniformly summable* (II) *with respect to u in an interval (a, b), with Sum $S(u)$, and if the terms $a_v(u)$ are continuous functions of u in (a, b), then $S(u)$ is a continuous function of u in (a, b).*

Put

$$\sum_{v=0}^{n} a_v(u) f_v(x) = S_{n,x}(u),$$

and

$$\lim_{n \to \infty} S_{n,x}(u) = S_x(u),$$

then

$$\lim_{x \to \infty} S_x(u) = S(u).$$

We have

$$\begin{aligned} |S(u+h) - S(u)| \leqq {} & |S(u+h) - S_x(u+h)| + |S_x(u+h) \\ (a) \qquad & - S_{n,x}(u+h)| + |S_{n,x}(u+h) - S_{n,x}(u)| \\ & + |S_{n,x}(u) - S_x(u)| + |S_x(u) - S(u)|. \end{aligned}$$

From the definition of uniform summability (II), it follows that, for any given ϵ, we can find X such that for $x > X$, we have

$$(b) \qquad | S_x(u) - S(u) | < \frac{\epsilon}{5},$$

and

$$(c) \qquad | S(u+h) - S_x(u+h) | < \frac{\epsilon}{5},$$

and that we can find N such that for $n > N$,

$$(d) \qquad | S_{n,x}(u) - S_x(u) | < \frac{\epsilon}{5},$$

and

$$(e) \qquad | S_x(u+h) - S_{n,x}(u+h) | < \frac{\epsilon}{5}.$$

Since $S_{n,x}(u)$ is a sum of a finite number of continuous functions, we can find δ such that for $|h| < \delta$, we have

$$(f) \qquad | S_{n,x}(u+h) - S_{n,x}(u) | < \frac{\epsilon}{5}.$$

Combining (b)–(f), we get

$$| S(u+h) - S(u) | < \epsilon \qquad \text{for } |h| < \delta,$$

from which the theorem follows.

24. THEOREM 13. *If* $\sum_0^\infty a_v(u)$ *is uniformly summable* (II) *with respect to u in an interval (a, b), with Sum $S(u)$, and if its terms $a_v(u)$ are integrable, then the series obtained by integrating the given series term by term with respect to u over a range (c_1, c_2) included in (a, b) is summable* (II) *with Sum* $\int_{c_1}^{c_2} S(u)du$.

Using the notation of § 23, since

$$\lim_{n\to\infty} S_{n,x}(u) = S_x(u)$$

and the approach is uniform with respect to u in (a, b), and since

$$\lim_{n\to\infty} S_{n,x}(u) = \sum_0^\infty a_v(u) f_v(x),$$

it follows that the series

$$S_x(u) = \sum_0^\infty a_v(u) f_v(x)$$

is uniformly convergent in (a, b). Hence

$$\int_{c_1}^{c_2} S_x(u)\, du = \sum_0^\infty f_v(x) \int_{c_1}^{c_2} a_v(u)\, du.$$

$$\therefore \lim_{x\to\infty}\lim_{n\to\infty}\sum_{v=0}^{n} f_v(x)\int_{c_1}^{c_2} a_v(u)du = \lim_{x\to\infty}\sum_{0}^{\infty} f_v(x)\int_{c_1}^{c_2} a_v(u)du = \lim_{x\to\infty}\int_{c_1}^{c_2} S_x(u)du.$$

Then, to prove our theorem, we must show that

$$\lim_{x\to\infty}\int_{c_1}^{c_2} S_x(u)\,du = \int_{c_1}^{c_2} S(u)\,du.$$

We can write

$$S(u) = S_x(u) + \eta_x(u),$$

where $\eta_x(u)$ approaches its limit 0, as $x\to\infty$, uniformly with respect to u in (a, b); we can then find X such that for $x > X$,

$$|\eta_x(u)| < \epsilon \qquad \text{for every } u \text{ in } (a, b).$$

$$\int_{c_1}^{c_2} S(u)\,du = \int_{c_1}^{c_2} S_x(u)\,du + \int_{c_1}^{c_2} \eta_x(u)\,du.$$

But

$$\left|\int_{c_1}^{c_2} \eta_x(u)\,du\right| \leqq \int_{c_1}^{c_2} |\eta_x(u)|\cdot|du| < \epsilon(c_2 - c_1) \quad \text{for } x > X.$$

$$\therefore \lim_{x\to\infty}\int_{c_1}^{c_2} \eta_x(u)\,du = 0 \quad \text{and} \quad \lim_{x\to\infty}\int_{c_1}^{c_2} S_x(u)\,du = \int_{c_1}^{c_2} S(u)\,du.$$

25. THEOREM 14. *If* $\sum_{0}^{\infty} a_v(u)$ *is summable* (II) *with Sum* $S(u)$, *and if the series* $\sum_{0}^{\infty} a'_v(u)$ *obtained by differentiating the given series term by term with respect to* u *is uniformly summable* (II), *with Sum* $\sigma(u)$, *then*

$$\sigma(u) = \frac{d}{du} S(u).$$

The proof is practically the same as that of § 12.

Case III.

26. Let $f_v(n, p)$ be defined for all positive values of n, p, and for

$$v = 0, 1, \cdots, [n],$$

and let

$$f_v(n, p) = 0 \qquad \text{for } v > n.$$

Suppose that $f_v(n, p)$ satisfies the following conditions:

1° $\qquad 0 \leqq f_v(n, p) \leqq 1 \qquad$ for every v, n, p;

2° when n, p are fixed, the sequence $f_0, f_1, \cdots, f_v, \cdots$ is decreasing;

3° $\lim_{p\to\infty} f_v(n, p) = 1$, and when N has been fixed, we can choose n_0 so that

$f_v(n, p) \to 1$ uniformly for

$$v = 0, 1, \cdots, N, n \geqq n_0;$$

4° $$\lim_{p\to\infty}\lim_{n\to\infty} f_v(n, p) = 1 \qquad \text{for } v \text{ fixed;}$$

5° $$\lim_{\substack{n, p\to\infty \\ F}} f_v(n, p) = 1 \quad \text{for } v \text{ fixed, for certain paths } F;$$

6° $$\lim_{p\to\infty}\lim_{n\to\infty} f_{[n]}(n, p) = 0;$$

7° $$\lim_{\substack{n, p\to\infty \\ F}} f_{[n]}(n, p) = 0 \qquad \text{for certain paths } F.$$

Hardy and Chapman* have shown that the limits (7) and (9) can only exist when the given series Σa_v is convergent, so that they will not give rise to methods of summation of non-convergent series. We need not consider them further.

When the limit (8) exists and is equal to S:

(8′) $$\lim_{p\to\infty}\lim_{n\to\infty}\sum_{v=0}^{[n]} a_v f_v(n, p) = S,$$

we shall say that the series Σa_v is *summable* (*III A*) with Sum S.

When the limit (10) exists, with the value S:

(10′) $$\lim_{\substack{n, p\to\infty \\ F}}\sum_{v=0}^{[n]} a_v f_v(n, p) = S,$$

we shall say that the series Σa_v is *summable* (*III B*) with Sum S.

We shall now show that both these methods satisfy the conditions of consistency.

27. Theorem 15. *If the series* $\sum_0^\infty a_v$ *is convergent with sum* s, *then it is summable* (III A) *and also summable* (III B) *with Sum equal to* s *in both cases, so that part* (1) *of the conditions of consistency is satisfied for both methods.*

Hardy and Chapman have proved† that when Σa_v is convergent with sum s, then

$$\lim_{n, p\to\infty}\sum_{v=0}^{[n]} a_v f_v(n, p) = s.$$

It follows at once that the limit (10) taken along any path F will be equal to s.

Since $f_v(n, p) \gtreqless 1$, the series $\sum_0^\infty a_v f_v(n, p)$ is convergent, so that $\lim_{n\to\infty}\sum_{v=0}^{[n]} a_v f_v(n, p)$ exists. Then from the theorem on double limits quoted in § 14, we see that the limit (8) exists and is equal to s.

* *Quarterly Journ. of Math.*, Vol. 42 (1911), p. 202.

† *Quarterly Journ. of Math.*, Vol. 42 (1911), p. 202.

28. THEOREM 16. *If the series* $\sum_{0}^{\infty} a_v(u)$ *is uniformly convergent, with sum* $s(u)$, *in an interval* (a, b), *then it is also uniformly summable* (III A) *and* (III B) *with respect to* u *in* (a, b), *and part* (3) *of the conditions of consistency is satisfied for both methods.*

We shall first prove that $\sum_{v=0}^{[n]} a_v(u) f_v(n, p)$ approaches its Pringsheim double limit $s(u)$ uniformly with respect to u in (a, b).

If we put

$$1 - f_v(n, p) = g_v(n, p),$$

and

$$G(n, p, u) = \sum_{v=0}^{[n]} a_v(u)\, g_v(n, p),$$

we must prove that numbers N, P can be found such that for $n > N$ and $p > P$, we have

$$|G(n, p, u)| < \epsilon$$

for every u in (a, b).

Just as in § 22, we have

$$\left| \sum_{v=N+1}^{[n]} a_v(u)\, g_v(n, p) \right| \overline{\gtrless} A(u) \cdot g \overline{\gtrless} A(u),$$

for every p and u, where $A(u)$ and g are defined as before. Since $\sum_{0}^{\infty} a_v(u)$ is uniformly convergent, we can find N_0 such that $A(u) < \epsilon / 2$, and therefore

$$(a) \qquad \left| \sum_{N+1}^{[n]} a_v(u)\, g_v(n, p) \right| < \frac{\epsilon}{2}$$

for $N > N_0$, for every p, and for every u in (a, b).

Now when N is fixed, by condition 3° we can choose n_0, p_0 such that $|g_v(n, p)| < \epsilon'$ for $v = 0, 1, \cdots, N$, $n > n_0$, $p > p_0$; hence

$$\left| \sum_{v=0}^{N} a_v(u)\, g_v(n, p) \right| < \epsilon' \sum_{0}^{N} |a_v(u)| \quad (n > n_0, p > p_0).$$

Let K be the upper bound of $\sum_{0}^{N} |a_v(u)|$ for u in (a, b), and take $\epsilon' < \epsilon / 2K$. Then

$$(b) \qquad \left| \sum_{v=0}^{N} a_v(u)\, g_v(n, p) \right| < \frac{\epsilon}{2}$$

for $n > n_0$, $p > p_0$, and for every u in (a, b).

(*a*) and (*b*) then give

$$(c) \qquad |G(n, p, u)| < \epsilon$$

for $n > N'$, $p > p_0$, and for every u in (a, b), where N' is the greater of N_0 and n_0.

It follows at once from the result just proved that $\sum_{v=0}^{[n]} a_v(u) f_v(n, p)$ approaches its double limit along a path F:

$$\lim_{\substack{n, p \to \infty \\ F}} \sum_{v=0}^{[n]} a_v(u) f_v(n, p) = s(u)$$

uniformly with respect to u in (a, b); so that the part of our theorem which relates to the method (III B) is proved.

Using Theorem 10, we can readily show, as in § 22, that as $n \to \infty$, $\sum_{v=0}^{[n]} a_v(u) f_v(n, p)$ approaches a limit uniformly with respect to u in (a, b). Then by Theorem 9, it follows from the result obtained in the first part of this section, that $\sum_{v=0}^{[n]} a_v(u) f_v(n, p)$ approaches its repeated double limit

$$\lim_{p \to \infty} \lim_{n \to \infty} \sum_{v=0}^{[n]} a_v(u) f_v(n, p) = s(u)$$

uniformly with respect to u in (a, b); so that the part of our theorem which refers to the method (III A) is proved.

29. Theorem 17. *If the series* $\sum_0^\infty a_v$ *is properly divergent, with* $\lim_{n \to \infty} s_n = +\infty$, *then it is not summable* (III A) or (III B) *with finite Sum; hence part* (2) *of the conditions of consistency is satisfied.*

Equation (a) of § 9 becomes in this case:

$$(a) \quad \sum_{v=0}^{[n]} a_v f_v(n, p) = \sum_{v=0}^{m} (s_v - K) h_v(n, p) + K \{ f_0(n, p) - f_{[n]}(n, p) \}$$
$$+ \sum_{v=m+1}^{[n]-1} r_v h_v(n, p) + s_{[n]} f_{[n]}(n, p),$$

where $h_v(n, p) = f_v(n, p) - f_{v+1}(n, p)$.

From conditions 4°, 6°, we have

$$\lim_{p \to \infty} \lim_{n \to \infty} h_v(n, p) = 0, \quad \lim_{p \to \infty} \lim_{n \to \infty} f_0(n, p) = 1, \quad \lim_{p \to \infty} \lim_{n \to \infty} f_{[n]}(n, p) = 0.$$

Hence

$$(b) \quad \lim_{p \to \infty} \lim_{n \to \infty} \sum_{v=0}^{[n]} a_v f_v(n, p) = K + \lim_{p \to \infty} \lim_{n \to \infty} \sum_{v=m+1}^{[n]-1} r_v h_v(n, p)$$
$$+ \lim_{p \to \infty} \lim_{n \to \infty} s_{[n]} f_{[n]}(n, p).$$

But since $r_v > 0$, $h_v(n, p) > 0$ (by 2°), $s_{[n]} > 0$, $f_{[n]}(n, p) > 0$, the last

two terms on the right in (b) are positive, so that

$$\lim_{p\to\infty}\lim_{n\to\infty}\sum_{v=0}^{[n]} a_v f_v(n, p) > K.$$

Hence Σa_v is not summable (III A) with finite Sum.

Again, from conditions 5°, 7°, we have

$$\lim_{\substack{n, p\to\infty\\ F}} h_v(n, p) = 0, \quad \lim_{\substack{n, p\to\infty\\ F}} f_0(n, p) = 1, \quad \lim_{\substack{n, p\to\infty\\ F}} f_{[n]}(n, p) = 0.$$

Also

$$\lim_{\substack{n, p\to\infty\\ F}} \sum_{v=m+1}^{[n]-1} r_v h_v(n, p) > 0, \quad \lim_{\substack{n, p\to\infty\\ F}} s_{[n]} f_{[n]}(n, p) > 0;$$

hence

$$\lim_{\substack{n, p\to\infty\\ F}} \sum_{v=0}^{[n]} a_v f_v(n, p) > K,$$

and therefore Σa_v is not summable (III B) with finite Sum.

30. Theorem 18. *If* $\sum_0^\infty a_v(u)$ *is uniformly summable* (III A) *or* (III B) *with respect to* u *in an interval* (a, b) *with Sum* $S(u)$, *and if the terms* $a_v(u)$ *are continuous functions of* u *in* (a, b), *then* $S(u)$ *is continuous in* (a, b).

The proof of the part of the theorem relating to the method (III A) is precisely similar to that of Theorem 12.

To prove the other part, put

$$\sum_{v=0}^{[n]} a_v(u) f_v(u, p) = S_{n, p}(u),$$

then $S_{n, p}(u)$ approaches its limit

$$\lim_{\substack{n, p\to\infty\\ F}} S_{n, p}(u) = S(u)$$

uniformly with respect to u in (a, b), so that

$$(a) \qquad | S(u) - S_{n, p}(u) | < \frac{\epsilon}{3}$$

for every u in (a, b), if n, p are so chosen (satisfying the relation $F(n, p)=0$) that the corresponding point (n, p) is sufficiently far along the path F.

Writing

$$| S(u+h) - S(u) | \overline{\gtrless} | S(u+h) - S_{n, p}(u+h) | + | S_{n, p}(u+h) - S_{n, p}(u) | + | S_{n, p}(u) - S(u) |,$$

the first and last terms on the right are, by (a), each $< \epsilon / 3$ for proper choice

of n, p, and the second term can be made $< \epsilon/3$ by taking $|h| < \delta$; hence

$$|S(u+h) - S(u)| < \epsilon \qquad \text{for } |h| < \delta,$$

and our theorem follows.

31. THEOREM 19. *If the series* $\sum_0^\infty a_v(u)$ *is uniformly summable* (III A) *or* (III B) *with respect to* u *in* (a, b), *with Sum* $S(u)$, *and if the terms* $a_v(u)$ *are integrable, then the series obtained by integrating the given series term by term with respect to* u *over a range* (c_1, c_2) *included in* (a, b) *is summable* (III A) *or* (III B) *respectively, with Sum* $\int_{c_1}^{c_2} S(u)\, du$.

The proof of the part referring to (III A) is the same as that of Theorem 13. To prove the other part, let us write

$$S(u) = S_{n,p}(u) + \delta_{n,p}(u),$$

where $\delta_{n,p}(u)$ approaches its limit

$$\lim_{\substack{n,p \to \infty \\ F}} \delta_{n,p}(u) = 0$$

uniformly with respect to u in (a, b).

(a) $$\therefore \int_{c_1}^{c_2} S(u)\, du = \int_{c_1}^{c_2} S_{n,p}(u)\, du + \int_{c_1}^{c_2} \delta_{n,p}(u)\, du,$$

and

(b) $$\int_{c_1}^{c_2} S_{n,p}(u)\, du = \sum_{v=0}^{[n]} f_v(n,p) \int_{c_1}^{c_2} a_v(u)\, du.$$

By properly choosing n, p, we can make

$$|\delta_{n,p}(u)| < \epsilon$$

for every u in (a, b);

$$\left| \int_{c_1}^{c_2} \delta_{n,p}(u)\, du \right| \overline{\gtrless} \int_{c_1}^{c_2} |\delta_{n,p}(u)|\, |du| < \epsilon (c_2 - c_1)$$

for n, p properly chosen.

$$\therefore \lim_{\substack{n,p \to \infty \\ F}} \int_{c_1}^{c_2} \delta_{n,p}(u) du = 0,$$

and

$$\lim_{\substack{n,p \to \infty \\ F}} \sum_{v=0}^{[n]} f_v(n,p) \int_{c_1}^{c_2} a_v(u)\, du = \int_{c_1}^{c_2} S(u)\, du.$$

32. THEOREM 20. *If* $\sum_0^\infty a_v(u)$ *is summable* (III A) *or* (III B) *with Sum* $S(u)$, *and if the series* $\sum_0^\infty a_v'(u)$ *obtained by term by term differentiation of the*

given series is uniformly summable (III A) *or* (III B) *respectively, with Sum* $\sigma(u)$, *then*

$$\sigma(u) = \frac{d}{du} S(u).$$

The proof is the same as that of § 12.

Case IV.

33. Let $f_v(x)$ be defined for all positive values of x and for all positive integral values of v, including 0.

Suppose also that $f_v(x)$ satisfies the following conditions:

1° $$f_v(x) > 0 \qquad \text{for every } v, x;$$

2° $$\lim_{x\to\infty} f_v(x) = 0 \qquad \text{for } v \text{ fixed};$$

3° $\sum_0^\infty f_v(x)$ is convergent for every x, and

$$\lim_{x\to\infty} \sum_{v=0}^{\infty} f_v(x) = 1.$$

The limit (12) cannot be used for the summation of non-convergent series (nor even for convergent series), for by 2°,

$$\lim_{x\to\infty} \sum_{v=0}^{n} s_v f_v(x) = 0,$$

and therefore

(12′) $$\lim_{n\to\infty} \lim_{x\to\infty} \sum_{v=0}^{n} s_v f_v(x) = 0.$$

When the limit (13) exists and is equal to S,

(13′) $$\lim_{x\to\infty} \lim_{n\to\infty} \sum_{v=0}^{n} s_v f_v(x) = S,$$

we shall say that the series $\Sigma\, a_v$ is *summable* (IV) with Sum S.

34. THEOREM 21. *If the series* $\sum_0^\infty a_v$ *is convergent, with sum* s, *then*

$$\lim_{x\to\infty} \lim_{n\to\infty} \sum_{v=0}^{n} s_v f_v(x) = s;$$

that is, every convergent series is summable (IV) *with Sum equal to sum, so thta part* (1) *of the conditions of consistency is satisfied.*

Put

$$s_n = s + \delta_n, \qquad \text{where} \qquad \lim_{n\to\infty} \delta_n = 0.$$

Then

$$\sum_{v=0}^{n} s_v f_v(x) = s \cdot \sum_{v=0}^{n} f_v(x) + \sum_{v=0}^{n} \delta_v f_v(x).$$

By 3°,

$$\lim_{x\to\infty} \lim_{n\to\infty} \sum_{v=0}^{n} f_v(x) = 1,$$

so that

$$\lim_{x\to\infty} \lim_{n\to\infty} \sum_{v=0}^{n} s_v f_v(x) = s + \lim_{x\to\infty} \lim_{n\to\infty} \sum_{v=0}^{n} \delta_v f_v(x).$$

Since

$$\lim_{n\to\infty} \delta_n = 0,$$

we can find m such that $|\delta_v| < \epsilon$ for $v > m$.

Now

$$\left|\sum_{v=0}^{n} \delta_v f_v(x)\right| \gtreqless \sum_{v=0}^{n} |\delta_v| \cdot f_v(x),$$

$$\gtreqless \sum_{v=0}^{m} |\delta_v| \cdot f_v(x) + \sum_{v=m+1}^{n} |\delta_v| \cdot f_v(x),$$

and

$$\sum_{v=m+1}^{n} |\delta_v| \cdot f_v(x) < \epsilon \sum_{v=m+1}^{n} f_v(x) < \epsilon \sum_{0}^{n} f_v(x),$$

$$\therefore \sum_{v=0}^{\infty} |\delta_v| f_v(x) < \sum_{v=0}^{m} |\delta_v| f_v(x) + \epsilon \sum_{0}^{\infty} f_v(x).$$

Now since

$$\lim_{x\to\infty} f_v(x) = 0$$

for v finite,

$$\lim_{x\to\infty} \sum_{0}^{\infty} f_v(x) = 1,$$

and since ϵ can be taken as small as we please, we have

$$\lim_{x\to\infty} \lim_{n\to\infty} \sum_{v=0}^{n} \delta_v f_v(x) = 0.$$

We have then

$$\lim_{x\to\infty} \lim_{n\to\infty} \sum_{v=0}^{n} s_v f_v(x) = s.$$

35. Theorem 22. *If* $\sum_{0}^{\infty} a_v$ *is properly divergent, so that*

$$\lim_{x\to\infty} s_n = +\infty,$$

then

$$\lim_{x\to\infty} \lim_{n\to\infty} \sum_{v=0}^{n} s_v f_v(x) = +\infty;$$

then part (2) of the conditions of consistency is satisfied.

As before, we can put $s_v = K + r_v$ for $v > m$, $r_v > 0$, where K is an arbitrarily large number. Then

$$\sum_{v=0}^{n} s_v f_v(x) = \sum_{v=0}^{m} s_v f_v(x) + K \sum_{v=m+1}^{n} f_v(x) + \sum_{v=m+1}^{n} r_v f_v(x)$$

$$= \sum_{v=0}^{m} (s_v - K) f_v(x) + K \sum_{v=0}^{n} f_v(x) + \sum_{v=m+1}^{n} r_v f_v(x).$$

Since

$$\lim_{x \to \infty} f_v(x) = 0, \quad \lim_{x \to \infty} \lim_{n \to \infty} \sum_{0}^{n} f_v(x) = 1, \quad r_v > 0, \quad f_v(x) > 0,$$

we have

$$\lim_{x \to \infty} \lim_{n \to \infty} \sum_{v=0}^{n} s_v f_v(x) = K + \lim_{x \to \infty} \lim_{n \to \infty} \sum_{v=m+1}^{n} r_v f_v(x) > K.$$

36. THEOREM 23. *If the series* $\sum_{0}^{\infty} a_v(u)$ *is uniformly convergent in a closed interval* (a, b), *with sum* $s(u)$, *then* $\sum_{v=0}^{n} s_v(u) f_v(x)$ *approaches its repeated double limit*

$$\lim_{x \to \infty} \lim_{n \to \infty} \sum_{v=0}^{n} s_v(u) f_v(x) = s(u)$$

uniformly with respect to u *in* (a, b); *that is, a uniformly convergent series is also uniformly summable* (IV), *and part* (3) *of the conditions of consistency is satisfied.*

Write

$$s_n(u) = s(u) + \delta_n(u),$$

then we can find m such that for $n > m$,

(*a*) $$|\delta_n(u)| < \epsilon$$

for every u in (a, b).

(*b*) $$\sum_{v=0}^{n} s_v(u) f_v(x) = s(u) \sum_{v=0}^{n} f_v(x) + \sum_{v=0}^{n} \delta_v(u) f_v(x).$$

Now by 3°, $\sum_{0}^{\infty} f_v(x)$ is convergent for all values of x, so that if we denote its sum by $F(x)$, we can find N such that for $n > N$ and for x fixed,

$$\left| F(x) - \sum_{0}^{n} f_v(x) \right| < \epsilon.$$

Then for any fixed u in (a, b), we have

$$\left| F(x) \cdot s(u) - s(u) \sum_{0}^{n} f_v(x) \right| < \epsilon |s(u)| \qquad (n > N).$$

Now since $s(u)$ is the sum of a series uniformly convergent in the closed interval (a, b), $|s(u)|$ must have a finite upper bound, say K, in (a, b); hence

$$\left| F(x) \cdot s(u) - s(u) \sum_0^n f_v(x) \right| < K \cdot \epsilon$$

for $n > N$ and for every u in (a, b). Therefore $s(u) \sum_0^n f_v(x)$ approaches its limit $s(u) \cdot F(x)$ uniformly with respect to u in (a, b) as $n \to \infty$.

Again, since by 3°, $\lim_{x \to \infty} F(x) = 1$, we can find X such that for $x > X$,

$$|F(x) - 1| < \epsilon.$$

For any fixed u in (a, b), we then have

$$|F(x) \cdot s(u) - s(u)| < \epsilon |s(u)| \qquad \text{for } x > X;$$

therefore

$$|F(x) s(u) - s(u)| < K \epsilon$$

for $x > X$ and for every u in (a, b). Hence $F(x) s(u)$ approaches its limit $s(u)$ uniformly with respect to u in (a, b) as $x \to \infty$.

Referring to our definition (§ 20), we see that $s(u) \sum_0^n f_v(x)$ approaches its repeated double limit

$$\lim_{x \to \infty} \lim_{n \to \infty} s(u) \sum_0^n f_v(x) = s(u)$$

uniformly with respect to u in (a, b).

We have next to show that $\sum_0^n \delta_v(u) f_v(x)$ approaches its repeated double limit

$$\lim_{x \to \infty} \lim_{n \to \infty} \sum_0^n \delta_v(u) f_v(x) = 0$$

uniformly with respect to u in (a, b).

We have

$$\left| \sum_0^n \delta_v(u) f_v(x) - \sum_0^m \delta_v(u) f_v(x) \right| \leqq \sum_{m+1}^n |\delta_v(u)| f_v(x).$$

By (a),

$$\sum_{m+1}^n |\delta_v(u)| f_v(x) < \epsilon \sum_{m+1}^n f_v(x) < \epsilon \sum_0^n f_v(x) < \epsilon \sum_0^\infty f_v(x)$$

for every u in (a, b). But $\sum_0^\infty f_v(x)$ is convergent for every x, so that

$$F(x) \equiv \sum_0^\infty f_v(x)$$

is bounded for all values of x, and if G be the upper bound of $F(x)$, we have

$$\sum_{m+1}^{n} |\delta_v(u)| f_v(x) < G\epsilon$$

for every u in (a, b).

$$\therefore \left| \sum_{v=0}^{n} \delta_v(u) f_v(x) - \sum_{v=0}^{m} \delta_v(u) f_v(x) \right| < G\epsilon$$

for every u in (a, b). Hence $\sum_{0}^{n} \delta_v(u) f_v(x)$ approaches its limit, as $n \to \infty$, uniformly with respect to u in (a, b).

We must now show that $\sum_{0}^{m} \delta_v(u) f_v(x)$ approaches its limit 0, as $x \to \infty$, uniformly with respect to u in (a, b). m is now fixed. Since

$$\lim_{x \to \infty} f_v(x) = 0,$$

for every v, we can find X_v such that for $x > X_v$, we have $f_v(x) < \epsilon$; let X be the greatest of the finite set $X_0, X_1, \cdots, X_m$, then we have each $f_v(x) < \epsilon$ for $x > X$ $(v = 0, 1, \cdots, m)$.

$$\therefore \left| \sum_{0}^{m} \delta_v(u) f_v(x) \right| \leqq \sum_{0}^{m} |\delta_v(u)| f_v(x) < \epsilon \sum_{0}^{m} |\delta_v(u)| \quad \text{for } x > X.$$

Now $\delta_v(u)$ is bounded in (a, b), so that $\sum_{0}^{m} |\delta_v(u)|$ has a finite upper bound, call it H. Then

$$\left| \sum_{0}^{m} \delta_v(u) f_v(x) \right| < H\epsilon$$

for $x > X$ and for every u in (a, b). Hence $\sum_{0}^{m} \delta_v(u) f_v(x)$ approaches its limit 0, as $x \to \infty$, uniformly with respect to u in (a, b).

We have now shown that $\sum_{0}^{n} \delta_v(u) f_v(x)$ approaches its repeated double limit 0 uniformly; and our theorem is now proved.

37. THEOREM 24. *If* $\sum_{0}^{\infty} a_v(u)$ *is uniformly summable* (IV) *with respect to* u *in an interval* (a, b), *with Sum* $S(u)$, *and if its terms* $a_v(u)$ *are continuous in* (a, b) *then* $S(u)$ *is continuous in* (a, b).

The proof of this theorem can be carried through in the same way as that of § 23, the only difference being that here $S_{n,x}(u)$ is expressed in terms of s_v instead of a_v, but since $s_v(u)$ is a sum of a finite number of continuous functions, it follows that $S_{n,x}(u)$ is here a continuous function of u.

38. THEOREM 25. *If* $\sum\limits_0^\infty a_v(u)$ *is uniformly summable* (IV) *with respect to* u *in* (a, b), *with Sum* $S(u)$, *and if the terms* $a_v(u)$ *are integrable, then the series obtained by integrating term by term the given series over a range* (c_1, c_2) *included in* (a, b), *is summable* (IV) *with Sum* $\int_{c_1}^{c_2} S(u)\, du$.

Putting

$$S_{n,x}(u) = \sum_0^n s_v(u) f_v(x), \qquad \lim_{n \to \infty} S_{n,x}(u) = S_x(u),$$

we have

$$S_x(u) = \lim_{n \to \infty} \sum_0^n s_v(u) f_v(x) = \sum_0^\infty s_v(u) f_v(x),$$

so that the series $\sum\limits_0^\infty s_v(u) f_v(x)$ is uniformly convergent, and we can integrate term by term, and get

$$\sum_{v=0}^\infty f_v(x) \int_{c_1}^{c_2} s_v(u)\, du = \int_{c_1}^{c_2} S_x(u)\, du,$$

the series on the left being convergent. Hence

$$\lim_{x \to \infty} \lim_{n \to \infty} \sum_{v=0}^n f_v(x) \int_{c_1}^{c_2} s_v(u)\, du = \lim_{x \to \infty} \int_{c_1}^{c_2} S_x(u)\, du.$$

Since

$$\lim_{x \to \infty} S_x(u) = S(u)$$

and this approach is uniform with respect to u, by the method used in § 24 we can prove that

$$\lim_{x \to \infty} \int_{c_1}^{c_2} S_x(u)\, du = \int_{c_1}^{c_2} S(u)\, du.$$

$$(a) \qquad \therefore \lim_{x \to \infty} \lim_{n \to \infty} \sum_{v=0}^n f_v(x) \int_{c_1}^{c_2} s_v(u)\, du = \int_{c_1}^{c_2} S(u)\, du.$$

To prove that the series $\sum\limits_0^\infty \int_{c_1}^{c_2} a_v(u)\, du$ is summable (IV) with Sum $\int_{c_1}^{c_2} S(u)\, du$, we must show that

$$\lim_{x \to \infty} \lim_{n \to \infty} \sum_{v=0}^n \left\{ \sum_{\rho=0}^v \int_{c_1}^{c_2} a_v(u)\, du \right\} f_v(x) = \int_{c_1}^{c_2} S(u)\, du.$$

But

$$\sum_{\rho=0}^v \int_{c_1}^{c_2} a_v(u)\, du = \int_{c_1}^{c_2} \left\{ \sum_{\rho=0}^v a_v(u) \right\} du = \int_{c_1}^{c_2} s_v(u)\, du,$$

and hence by (*a*), our theorem follows at once.

39. THEOREM 26. *If* $\sum_{0}^{\infty} a_v(u)$ *is summable* (IV) *with Sum* $S(u)$, *and if the series* $\sum_{0}^{\infty} a'_v(u)$ *obtained by term by term differentiation of the given series is uniformly summable* (IV) *with Sum* $\sigma(u)$, *then*

$$\sigma(u) = \frac{d}{du} S(u).$$

The proof is precisely the same as that of Theorem 14.

40. Reviewing the general results obtained, we see that the conditions of consistency are satisfied by all four of our general methods of summation (I)–(IV); and that uniformly summable series possess properties similar to those of uniformly convergent series.

CHAPTER III.

Particular Methods.

A number of particular known methods of summation are included as special cases under our methods (I)–(IV), and we shall now apply our general theorems to these.

Case I.

Cesàro's Method.

41. Let the function $f_v(n, k)$ be defined for all positive values of n, for $v = 1, 2, \cdots, [n]$, and for every real k except negative integral values, by the equation

$$(15)\quad f_v(n, k) = \frac{n(n-1)(n-2)\cdots(n-v+1)}{(k+n)(k+n-1)(k+n-2)\cdots(k+n-v+1)},$$

and for $v = 0$, let $f_0(n, k) = 1$, for $v > n$, let $f_v(n, k) = 0$. We shall call this the Cesàro function, and denote it by $Cf_v(n, k)$.

We must first show that $Cf_v(n, k)$ satisfies the conditions of § 6.

It is easily seen that 3° and 4° are true for every k; also that 1° is true if $k > 0$, but if $k < 0$ and $n > |k|$, then

$$Cf_v(n, k) \gtreqless 1.$$

Since

$$\frac{Cf_{v+1}(n, k)}{Cf_v(n, k)} = \frac{n-v}{k+n-v},$$

this ratio is < 1 if $k > 0$, and is > 1 if $k < 0$ and n is sufficiently large, so that the sequence (Cf_v) is decreasing for $k > 0$ and increasing for $k < 0$ and n large enough; 2° is then true for $k > 0$. We shall not consider values of $k < 0$.

It remains only to show that 5° is satisfied. For Cesàro's method it is sufficient to give n only positive integral values. We need then only prove that

$$\lim_{n\to\infty} Cf_n(n, k) = 0.$$

We have*

$$\frac{n!\, n^k}{(k+n)(k+n-1)\cdots(k+1)} = \Gamma_n(k+1),$$

* See Nielsen, Lehrbuch der unendlichen Reihen, p. 248.

where

$$\lim_{n\to\infty} \Gamma_n (k+1) = \Gamma (k+1).$$

$$C f_n (n, k) = \frac{n!}{(k+n) \cdots (k+1)} = n^{-k} \Gamma_n (k+1).$$

$$\therefore \lim_{n\to\infty} C f_n (n, k) = 0.$$

When a series Σa_v is summable (I) with the Cesàro function $C f_v (n, k)$, it is usual to say that the series is *summable* (C, k).

The corresponding summation formula is

$$(16) \qquad S = \lim_{n\to\infty} \sum_{v=0}^{n} a_v \cdot \frac{n(n-1)\cdots(n-v+1)}{(k+n)(k+n-1)\cdots(k+n-v+1)}.^{*}$$

42. The Theorems 1–3 give us the following results:

THEOREM 27. *Every convergent series is summable* (C, k) *for every* $k > 0$, *with a Sum equal to its sum.*

This theorem has been proved by CHAPMAN.†

THEOREM 28. *A properly divergent series is not summable* (C, k) *with finite Sum for any value of* $k > 0$.

This theorem is new; it has been proved when k is a positive integer, however, by NIELSEN.‡

THEOREM 29. *A uniformly convergent series is uniformly summable* (C, k) *for* $k > 0$.

This has been proved by CHAPMAN.§

Riesz's Method.

43. Let the function $f_v (n, k)$ be defined for every positive value of n, for every real k, and for $v = 1, 2, \cdots, [n]$, by the equation

$$(17) \qquad f_v (n, k) = \left\{1 - \frac{\lambda(v)}{\lambda(n)}\right\}^k,$$

where $\lambda(n)$ is a positive monotonic function of n, increasing to ∞ with n; for $v = 0$, let

$$f_0 (n, k) = 1,$$

for this we must assume that $\lambda(0) = 0$; and for $v > n$, let

$$f_v (n, k) = 0.$$

We shall call this the Riesz function, and denote it by $R f_v (n, k)$.

* When k is a positive integer, this is the definition given by CESÀRO (*Bulletin des sciences math.*, ser. 2, Vol. 14 (1890), pp. 114–120). When k is any real number > -1, the method has been discussed by CHAPMAN (*Proc. London Math. Soc.*, ser. 2, Vol. 9 (1911), pp. 369–409), and by KNOPP (*Sitzungsberichte der Berliner Math. Gesell.*, Vol. 7 (1907), pp. 1–12).

† *Proc. Lond. Math. Soc.*, ser. 2, Vol. 9 (1911), p. 377.

‡ NIELSEN, Elemente der Funktionentheorie (1911), pp. 194–5.

§ *Quarterly Journ. of Math.*, Vol. 43, pp. 24–25.

It follows at once from the definition that conditions 1°–4° of § 6 are satisfied when $k > 0$.

We shall suppose further that $\lambda(n)$ satisfies the condition:

$$\lim_{n\to\infty} \frac{\lambda(n-1)}{\lambda(n)} = 1, \tag{18}$$

then it follows that

$$\lim_{n\to\infty} \frac{\lambda([n])}{\lambda(n)} = 1$$

and

$$\lim_{n\to\infty} \left\{1 - \frac{\lambda([n])}{\lambda(n)}\right\}^k = 0,$$

so that condition 5° of § 6 is satisfied for $k > 0$.

When a series is summable (I) with the Riesz function $Rf_v(n, k)$, HARDY and CHAPMAN call it *summable* (R, λ, k).* The corresponding summation formula is

$$S = \lim_{n\to\infty} \sum_{v=0}^{[n]} a_v \left\{1 - \frac{\lambda(v)}{\lambda(n)}\right\}^k. \tag{19}$$

44. The Theorems 1–3 applied to this method give:

THEOREM 30. *A convergent series is summable* (R, λ, k) *for every* $k > 0$, *with Sum equal to sum, if* λ *satisfies the condition* (18).

This is proved by CHAPMAN and RIESZ.

THEOREM 31. *A properly divergent series cannot be summable* (R, λ, k), *for any* $k > 0$, *with finite Sum, if* λ *satisfies* (18).

This result is new.

THEOREM 32. *A uniformly convergent series is uniformly summable* (R, λ, k) *for every* $k > 0$, *if* λ *satisfies* (18).

This theorem is also new.

Case II.

LeRoy's Method.

45. Let the function $f_v(t)$ be defined for all positive integral values of v, including 0, and for all values of t such that $0 < t < 1$, by the equation

$$f_v(t) = \frac{\Gamma(vt + 1)}{\Gamma(v + 1)}. \tag{20}$$

We shall consider it for values of t near 1. We call it the LeRoy function and denote it by $Lf_v(t)$.

This function is positive for all values of v, t under consideration, and since

* See CHAPMAN, *Proc. Lond. Math. Soc.*, ser. 2, Vol. 9 (1911), p. 373.

$\Gamma(x)$ increases as x increases, for $x > 1.462 \cdots$*, we have

$$Lf_v(t) \gtreqless 1.$$

Also

$$\lim_{t \to 1} Lf_v(t) = 1,$$

and

$$Lf_0(t) = 1.$$

That the sequence $Lf_0, Lf_1, \cdots, Lf_v, \cdots$ is decreasing may be shown as follows:

$$(a)\ Lf_{v+1}(t) - Lf_v(t) = \frac{\Gamma(vt+t+1)}{\Gamma(v+2)} - \frac{\Gamma(vt+1)}{\Gamma(v+1)}$$

$$= \frac{\Gamma(vt+t+1)}{(v+1)!} - \frac{\Gamma(vt+1)}{v!}$$

$$= \frac{1}{(v+1)!}\{\Gamma(vt+1+t) - (v+1)\Gamma(vt+1)\}.$$

Now

$$\Gamma(vt+1+t) = \int_0^\infty e^{-x} x^{vt+t}\, dx, \qquad \Gamma(vt+1) = \int_0^\infty e^{-x} x^{vt}\, dx.$$

By integration by parts, we get

$$\int e^{-x} x^{vt+t}\, dx = -e^{-x} x^{vt+t} + (vt+t)\int e^{-x} x^{vt+t-1}\, dx,$$

or

$$\int_0^\infty e^{-x} x^{vt+t}\, dx = t(v+1)\int_0^\infty e^{-x} x^{vt+t-1}\, dx.$$

$$\therefore \Gamma(vt+t+1) = t(v+1)\Gamma(vt+t) < (v+1)\Gamma(vt+t), \qquad \text{since } t<1.$$

But $\Gamma(vt+t) < \Gamma(vt+1)$, hence

$$(b)\qquad \Gamma(vt+t+1) - (v+1)\Gamma(vt+1) < 0.$$

From (*a*) and (*b*), we see that Lf_v decreases as v increases.

We shall next show that

$$\lim_{n \to \infty} \frac{\Gamma(nt+1)}{\Gamma(n+1)} = 0.$$

By STIRLING's formula,† we have

$$\lim_{n \to \infty} \frac{\Gamma(n+1)}{e^{-n} n^{n+\frac{1}{2}}\sqrt{2\pi}} = 1,$$

$$\therefore \lim_{n \to \infty} \frac{\Gamma(nt+1)}{e^{-nt}(nt)^{nt+\frac{1}{2}}\sqrt{2\pi}} \cdot \frac{e^{-n} n^{n+\frac{1}{2}}\sqrt{2\pi}}{\Gamma(n+1)} = 1,$$

* See the graph of $\Gamma(x)$ in KLEIN, Hypergeometrische Funktion, p. 122, or GODEFROY, Théorie élémentaire des séries, p. 250.

† See BROMWICH, Theory of Infinite Series, p. 462.

or

$$(a) \qquad \lim_{n\to\infty} \frac{\Gamma(nt+1)}{\Gamma(n+1)} \cdot \frac{e^{-n} n^n \cdot n^{\frac{1}{2}}}{e^{-nt} n^{nt} n^{\frac{1}{2}} t^{nt} \sqrt{t}} = 1 .$$

Now

$$\frac{e^{-n} n^n n^{\frac{1}{2}}}{e^{-nt} n^{nt} t^{nt} n^{\frac{1}{2}}} = e^{-n(1-t)} n^{n(1-t)} t^{-nt} = e^{-n(1-t)+n(1-t)\log n - nt \log t}$$

$$= e^{(1-t)n \log n - n[(1-t)+t\log t]}$$

But since $t < 1$,

$$(b) \qquad \lim_{n\to\infty} e^{(1-t)n \log n - n[1-t+t\log t]} = +\infty .$$

From (a) and (b), we see that we must have

$$\lim_{n\to\infty} \frac{\Gamma(nt+1)}{\Gamma(n+1)} = 0 .$$

Then

$$\lim_{t\to 1} \lim_{n\to\infty} Lf_n(t) = 0 .$$

If we now make the change of variable $t = e^{-1/x}$, we get a function of x:

$$Lf_v(x) = \frac{\Gamma(ve^{-1/x}+1)}{\Gamma(v+1)},$$

which satisfies all the conditions of § 13.

When a series is summable (II) with the LeRoy function $Lf_v(x)$, we shall say that the series is *summable* (L). The corresponding summation formula is

$$(21) \qquad S = \lim_{t\to 1} \lim_{n\to\infty} \sum_{v=0}^{n} \frac{a_v \Gamma(vt+1)}{\Gamma(v+1)} .$$

This definition is one used by LeRoy.*

46. When we apply the general Theorems 7, 8 to this method, we have the theorems:

Theorem 33. *A convergent series is summable* (L) *with Sum equal to sum.*

This was proved by Hardy.†

Theorem 34. *A properly divergent series is not summable* (L) *with finite Sum.*

This result is new.

Borel's Method.

47. Let the function $f_v(x)$ be defined for all positive values of x, and for every positive integral value of v, including 0, by the equation

$$(22) \qquad Bf_v(x) = \int_0^x e^{-\alpha} \frac{\alpha^v}{v!} d\alpha .$$

* *Annales de Toulouse*, ser. 2, Vol. 2 (1900), pp. 323–327.

† *Quarterly Journ. of Math.*, Vol. 35 (1903), pp. 36–37.

This function is always positive; and since the integrand is positive, we have

$$\int_0^x e^{-\alpha}\,\alpha^v\,d\alpha < \int_0^\infty e^{-\alpha}\,\alpha^v\,d\alpha = v\,!$$

so that $Bf_v(x) < 1$ for every v, x. Thus condition 1° of § 13 is satisfied.

By integration by parts, we find

$$\int e^{-\alpha}\,\alpha^{v+1}\,d\alpha = -\,e^{-\alpha}\,\alpha^{v+1} + (v+1)\int e^{-\alpha}\,\alpha^v\,d\alpha\,,$$

$$\therefore \frac{1}{(v+1)\,!}\int_0^\infty e^{-\alpha}\,\alpha^{v+1}\,d\alpha = -\,\frac{e^{-x}\,x^{v+1}}{(v+1)\,!} + \frac{1}{v\,!}\int_0^x e^{-\alpha}\,\alpha^v\,d\alpha\,,$$

or

$$Bf_{v+1}(x) - Bf_v(x) = -\,e^{-x}\frac{x^{v+1}}{(v+1)\,!}.$$

The right hand member of this equation is always negative, so that as v increases, $Bf_v(x)$ decreases, and 2° is satisfied.

We have at once

$$\lim_{x\to\infty} Bf_v(x) = 1\,,$$

and 3° is satisfied.

That 4° is satisfied, may be shown as follows:
We first seek the limit

$$\lim_{n\to\infty}\frac{\alpha^n}{n\,!}.$$

Stirling's formula gives us

$$\lim_{n\to\infty}\frac{n\,!}{e^{-n}\,n^{n+\frac{1}{2}}\sqrt{2\pi}} = 1\,.$$

Now

$$\frac{\alpha^n}{n\,!} = \frac{e^{-n}\,n^{n+\frac{1}{2}}\sqrt{2\pi}}{n\,!}\cdot\frac{\alpha^n}{e^{-n}\,n^{n+\frac{1}{2}}\sqrt{2\pi}}\,,$$

so that

(a) $$\lim_{n\to\infty}\frac{\alpha^n}{n\,!} = \lim_{n\to\infty}\frac{\alpha^n}{e^{-n}\,n^{n+\frac{1}{2}}\sqrt{2\pi}}\,.$$

We have

$$\frac{\alpha^n}{e^{-n}\,n^{n+\frac{1}{2}}} = e^n\,\alpha^n\,n^{-n-\frac{1}{2}} = e^{n+n\log\alpha-(n+\frac{1}{2})\log n} = e^{n[1+\log\alpha]-(n+\frac{1}{2})\log n}\,.$$

But

$$\lim_{n\to\infty}{}' e^{n(1+\log\alpha)-(n+\frac{1}{2})\log n} = 0\,,$$

(b) $$\therefore \lim_{n\to\infty}\frac{\alpha^n}{n\,!} = 0\,.$$

We can then find N such that $\alpha^n / n! < \epsilon$ for $n > N$.

$$\therefore \int_0^x e^{-\alpha}\frac{\alpha^n}{n!}d\alpha < \epsilon \int_0^x e^{-\alpha}\, d\alpha = \epsilon\,(1 - e^{-x}) \qquad \text{for } n > N.$$

$$\therefore \lim_{n\to\infty}\int_0^x e^{-\alpha}\frac{\alpha^n}{n!}d\alpha = 0,$$

and

$$\lim_{x\to\infty}\lim_{n\to\infty} Bf_v(x) = 0,$$

so that 4° is satisfied.

All the conditions of § 13 are therefore satisfied by Borel's function.

The summation formula for this method is

$$S = \lim_{x\to\infty}\lim_{n\to\infty}\sum_{v=0}^{n} a_v \cdot \int_0^x e^{-\alpha}\frac{\alpha^v}{v!}\,d\alpha$$

$$= \lim_{x\to\infty}\lim_{n\to\infty}\int_0^x e^{-\alpha}\left(\sum_{v=0}^{n} a_v\frac{\alpha^v}{v!}\right)d\alpha.$$

If the series $\sum_0^\infty a_v\,\alpha^v / v!$ defines an integral function, or if it defines an analytic function which is susceptible of analytic continuation along the real axis, and therefore has no singular points on the real axis, we get

$$S = \lim_{x\to\infty}\int_0^x e^{-\alpha}\left(\sum_0^\infty a_v\frac{\alpha^v}{v!}\right)d\alpha.$$

$$(23) \qquad \therefore S = \int_0^\infty e^{-x}\left(\sum_0^\infty a_v\frac{x^v}{v!}\right)dx.$$

This is Borel's famous integral definition.* When a series is summable (II) with Borel's function $f_v(x)$, that is, when the integral (23) is convergent, we say that the series is *summable* (B_2).

48. We find, by applying Theorems 7 and 8, the results:

Theorem 35. *A convergent series is summable* (B_2) *with Sum equal to sum.*

This was proved by Hardy.†

Theorem 36. *A properly divergent series is not summable* (B_2) *with finite Sum.*

This is proved in Bromwich, Theory of Infinite Series, p. 270.

Euler's Power Series Method.

49. The function

$$(24) \qquad f_v(x) = e^{-\frac{v}{x}}$$

evidently satisfies the conditions 1°–4° of § 13.

* See Borel, Leçons sur les séries divergentes, p. 99.

† *Cambridge Philos. Transactions*, Vol. 19, pp. 298–299.

Our summation formula here is

$$S = \lim_{x \to \infty} \lim_{n \to \infty} \sum_{0}^{n} a_v \, e^{-\frac{v}{x}}.$$

If we put $e^{-\frac{1}{x}} = z$, it becomes

$$S = \lim_{z \to 1} \sum_{0}^{\infty} a_v \, z^v. \tag{25}$$

This is sometimes called Euler's power series method.

Theorem 7 applied to this case gives Abel's theorem on the continuity of power series.

Case III.

The Cesàro-Riesz methods* appear here as a special case.

50. Let the function $f_v(n, p)$ be defined for all positive values of n, p, and for

$$v = 0, 1, 2, \cdots, [n],$$

by the equation

$$f_v(n, p) = \left\{1 - \frac{\lambda(v)}{\lambda(n)}\right\}^{\frac{\lambda(n)}{p}} \tag{26}$$

where $\lambda(n)$ is a positive monotonic function increasing to ∞ with n, and satisfying the conditions $\lambda(0) = 0$ and

$$\lim_{n \to \infty} \frac{\lambda(n-1)}{\lambda(n)} = 1.$$

It is easily seen that conditions 1°, 2°, and the first part of 3° of § 26 are satisfied. Hardy and Chapman† have shown that the second part of 3° is satisfied.

Since

$$\lim_{n \to \infty} \left\{1 - \frac{\lambda(v)}{\lambda(n)}\right\}^{\frac{\lambda(n)}{p}} = e^{-\frac{\lambda(v)}{p}}$$

we have

$$\lim_{p \to \infty} \lim_{n \to \infty} f_v(n, p) = 1,$$

so that 4° is satisfied.

Since

$$\lim_{n \to \infty} \frac{\lambda(n-1)}{\lambda(n)} = 1,$$

and $\lambda(n) \to \infty$ as $n \to \infty$, we have

$$\lim_{n \to \infty} \left\{1 - \frac{\lambda([n])}{\lambda(n)}\right\}^{\frac{\lambda(n)}{p}} = 0,$$

so that 6° is true.

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* Called thus by Hardy and Chapman, *Quarterly Journ. of Math.*, Vol. 42, p. 191.

† *Quarterly Journ. of Math.*, Vol. 42, p. 204.

Now suppose that the path F is defined by the relation

$$\frac{\lambda(n)}{p} = \omega(n), \text{ where } \lim_{n\to\infty} \omega(n) \text{ is finite;}$$

then

$$\lim_{\substack{n,p\to\infty \\ F}} \left\{1 - \frac{\lambda(v)}{\lambda(n)}\right\}^{\frac{\lambda(n)}{p}} = \lim_{n\to\infty} \left\{1 - \frac{\lambda(v)}{\lambda(n)}\right\}^{\omega(n)} = 1.$$

$$\lim_{\substack{n,p\to\infty \\ F}} \left\{1 - \frac{\lambda([n])}{\lambda(n)}\right\}^{\frac{\lambda(n)}{p}} = \lim_{n\to\infty} \left\{1 - \frac{\lambda([n])}{\lambda(n)}\right\}^{\omega(n)} = 0.$$

Thus 5° and 7° are also satisfied.

When a series is summable (III A) with the Cesàro-Riesz function, we shall call it *summable* (CR, λ), and when it is summable (III B) and the path F is given by the function $\omega(n)$, we shall call it *summable* (CR, λ, ω).

51. From Theorems 15 and 17, we have:

THEOREM 37. *A convergent series is summable* (CR, λ)* *with Sum equal to sum.*

This theorem is new.

THEOREM 38. *A convergent series is summable* (CR, λ, ω) *for every path* F *(as described in § 50), with Sum equal to sum.*

This was proved by HARDY and CHAPMAN.†

THEOREM 39. *A properly divergent series is not summable* (CR, λ) *nor* (CR, λ, ω).

This result is new.

Case IV.

52. Let us take first

$$f_v(x) = e^{-x} \cdot \frac{x^v}{v!}. \tag{27}$$

We have at once $f_v(x) > 0$,

$$\lim_{x\to\infty} f_v(x) = 0,$$

$$\sum_0^\infty f_v(x) = e^{-x} \sum_0^\infty \frac{x^v}{v!} = e^{-x} \cdot e^x = 1;$$

so that conditions 1°, 2°, 3°, of § 33 are all satisfied. The corresponding summation formula is

$$S = \lim_{x\to\infty} e^{-x} \sum_{v=0}^\infty s_v \frac{x^v}{v!}, \tag{28}$$

which is BOREL's exponential definition.‡

When a series is summable (IV) with BOREL's function (27), we shall say that it is *summable* (B_1).

* Where λ satisfies the conditions of § 50.

† *Quarterly Journ. of Math.*, Vol. 42, p. 204.

‡ See BOREL, Leçons sur les séries divergentes, p. 97.

More generally, take

$$(29) \qquad f_v(x) = e^{-x^k} \cdot \frac{x^{vk}}{v\,!},$$

where k is a positive integer; this can be shown to satisfy the conditions of § 33 just as for (27). When a series is summable (IV) with BOREL's function (29), we shall call it *summable* (B_3, k). When $k = 1$, we have summability (B_1). The summation formula is

$$(30) \qquad S = \lim_{x \to \infty} e^{-x^k} \sum_0^\infty s_v \frac{x^{v^k}}{v\,!}.$$

This is BOREL's generalization of his exponential method.*

If we take

$$(31) \qquad f_v(x) = e^{-e^x} \cdot \frac{e^{vx}}{v\,!},$$

it is easily shown that this function satisfies the conditions of § 33. The resulting method is one studied by COSTABEL.†

Still more generally, let $\phi(x)$ be an integral function $\sum_0^\infty c_n x^n$, where $c_n > 0$. Take

$$(32) \qquad f_v(x) = \frac{1}{\phi(x)} \cdot c_v x^v.$$

This function satisfies the conditions of § 33, and includes all the preceding functions of this §.

53. Applying Theorems 21, 22 to BOREL's methods, we have:

THEOREM 40. *A convergent series is summable* (B_1) *with Sum equal to sum.*

This is proved in VIVANTI-GUTZMER, Theorie der eindeutigen analytischen Funktionen, pp. 328–9.

THEOREM 41. *A properly divergent series is not summable* (B_1) *with finite Sum.*

This also is given in VIVANTI-GUTZMER, p. 329.

THEOREM 42. *A convergent series is summable* (B_3, k) *with Sum equal to sum for every* k.

This result is proved in BROMWICH, Theory of Infinite Series, pp. 300–301.

THEOREM 43. *A properly divergent series is not summable* (B_3, k) *with finite Sum, for any* k.

This is new.

54. The general theorems of Chapter II on the uniform summability of uniformly convergent series, and the continuity, integration, and differentiation by uniformly summable series apply of course to the particular methods of this chapter, but as all the results so obtained are new, they have not been stated explicitly.

* See BOREL, Leçons sur les séries divergentes, p. 129.

† *L'Enseignement mathématique*, Vol. 10 (1908), pp. 387–388.

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